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Cite as: AIP Conference Proceedings **2172**, 070004 (2019); https://doi.org/10.1063/1.5133540
Published Online: 13 November 2019

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Variability and Effect of Noise on the Corporate Dynamics of Coupled Oscillators

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Abstract.
A nonlinear dynamical model of two coupled neurons based on the Rulkov map is considered. Variability analysis of corporate dynamics depending on the type of activity of separated neurons and strength of coupling is performed. Transitions between stationary, periodic, quasiperiodic, and chaotic regimes of this neuron system are studied. Additional effects of random disturbances on this system are discussed. Noise-induced transitions between periodic and chaotic stochastic oscillations are demonstrated.

INTRODUCTION

Analysis of complex phenomena in systems composed of interacting nonlinear oscillators is an attractive problem of the modern science. Such systems are studied in the physics, engineering, biology and economics [1, 2, 3, 4].

New unexpected dynamical regimes can appear due to the cooperative effects. Even very simple dynamical systems being coupled exhibit transitions from equilibrium to periodic, quasiperiodic or chaotic behavior [5, 6, 7]. A mathematical analysis of these phenomena requires analysing the saddle-node, period-doubling, crisis, and Neimark-Sacker bifurcations [8, 9, 10]. A presence of inevitable noise can cause another, more complicated, regimes [11, 12, 13, 14, 15, 16].

In the present paper, we study a nonlinear dynamical model of two coupled neurons based on the Rulkov map [17]. In the case of isolated neuron, this map defines the following dynamical system

$$x_{t+1} = f(\gamma, x_t), \qquad f(\gamma, x) = \frac{\alpha}{1 + x^2} + \gamma, \quad \alpha = 4.1. \tag{1}$$

Here, x is a membrane voltage in a neuron and γ is a gating ion concentration. In Figure 1, attractors of this model are shown for $-2.5 < \gamma < 1$. As can be seen, in this interval, the one-dimensional simple map-based model (1) exhibits the Feigenbaum tree with a diversity of dynamical regimes: equilibria, discrete cycles, and chaotic attractors.

The aim of the present paper is to describe a variability of dynamics of the coupled system and to show how noise can drastically change its deterministic behavior.

ANALYSIS OF DETERMINISTIC DYNAMICS

Consider a system of two coupled neurons modeled by identical Rulkov maps:

$$\begin{aligned} x_{t+1} &= f(\gamma, x_t) + \sigma(y_t - x_t), \\ y_{t+1} &= f(\gamma, y_t) + \sigma(x_t - y_t). \end{aligned} \tag{2}$$

Proceedings of the 45th International Conference on Application of Mathematics in Engineering and Economics (AMEE'19)
AIP Conf. Proc. 2172, 070004-1–070004-8; https://doi.org/10.1063/1.5133540
Published by AIP Publishing. 978-0-7354-1919-3/$30.00

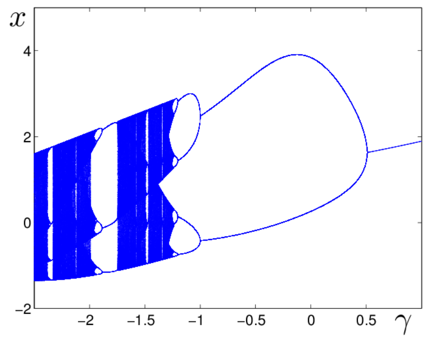

FIGURE 1. Bifurcation diagram of isolated neuron

Here, σ is a coupling parameter. This type of coupling is widely used in modeling the electrical synapse. A variation of the parameter γ allows to consider neurons with various types of the neural activity whereas the parameter σ controls the intensity of mutual feedbacks. An isolated neuron can exhibit both regular (equilibria or discrete cycles) and chaotic dynamics. We are interested in results of corporate dynamics: how the increase of the coupling parameter changes the system dynamics.

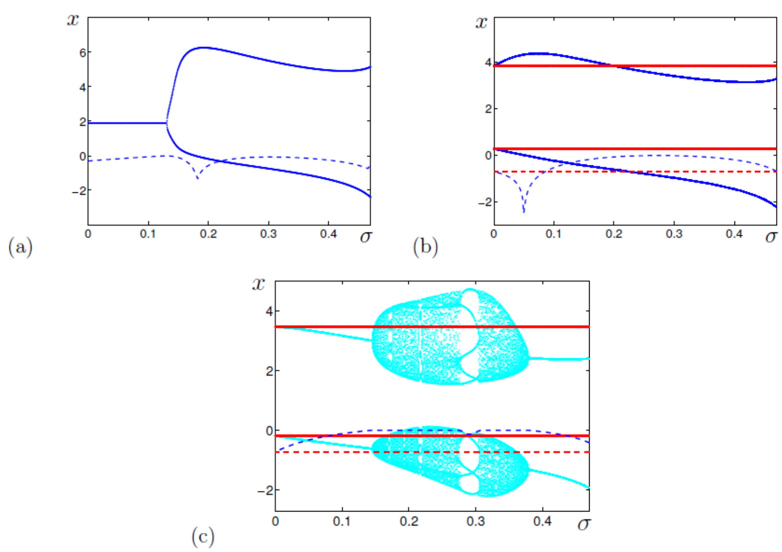

FIGURE 2. Bifurcation diagrams and Lyapunov exponents of two coupled neurons: (a) for $\gamma = 1$, (b) for $\gamma = 0$, (c) for $\gamma = -0.5$. Attractors are shown by solid lines, and Lyapunov exponents are shown by dashed lines.

First consider a case when isolated neurons are in equilibrium modes corresponding to the quiescence. In Figure 2a, attractors of system (2) with $\gamma = 1$ are shown by blue solid lines versus the coupling σ. Analysing the stability of the system (2) equilibrium $\bar{x} = \bar{y}$, one can find the bifurcation point $\sigma^* = (f_x'(\gamma, \bar{x})+1)/2, \quad f_x'(\gamma, x) = -2\alpha x/(1+x^2)^2$. For $\alpha = 4.1, \ \gamma = 1$, we have $\sigma^* = 0.131$. As the parameter σ passes σ^* from the left to right, the equilibrium loses its stability, and 2-cycle is born. So, the increasing coupling transforms this neuron system from the quiescence to the oscillatory regime. Here, we also show by dashed line a plot of the largest Lyapunov exponent $\Lambda(\sigma)$.

In Figure 2b for $\gamma = 0$, we consider a case when isolated neurons oscillate and exhibit the stable 2-cycle C_1. With the increase of σ, another cycle C_2 is split from C_1. So, for $\sigma \neq 0$, the system (2) becomes bistable, and the cycle C_2 (blue) changes with σ whereas the cycle C_1 (red) does not change.

An interesting phenomenon is observed for the smaller γ in the zone of 2-cycles (see Figure 2c for $\gamma = -0.5$). Here, the cycle C_1(red) does not change, but the cycle C_2 (light blue) undergoes the Neimark-Sacker bifurcation

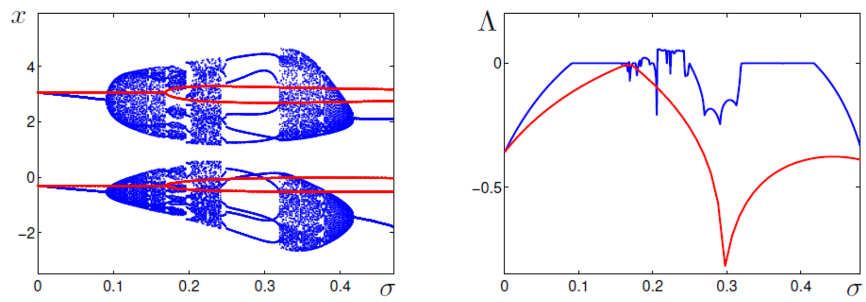

FIGURE 3. Bifurcation diagrams (left) and Lyapunov exponents (right) of two coupled neurons for $\gamma = -0.7$.

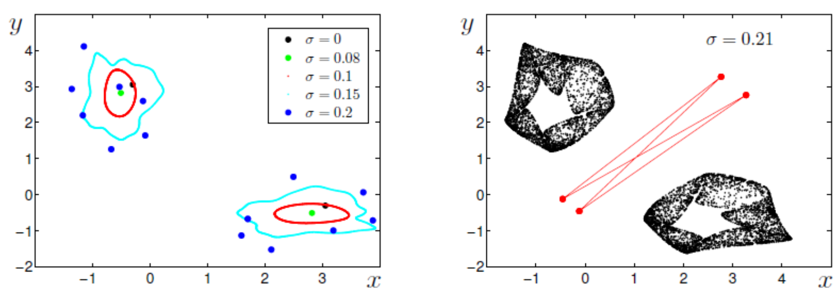

FIGURE 4. Attractors of system (2) with $\gamma = -0.7$ for various values of the coupling parameter.

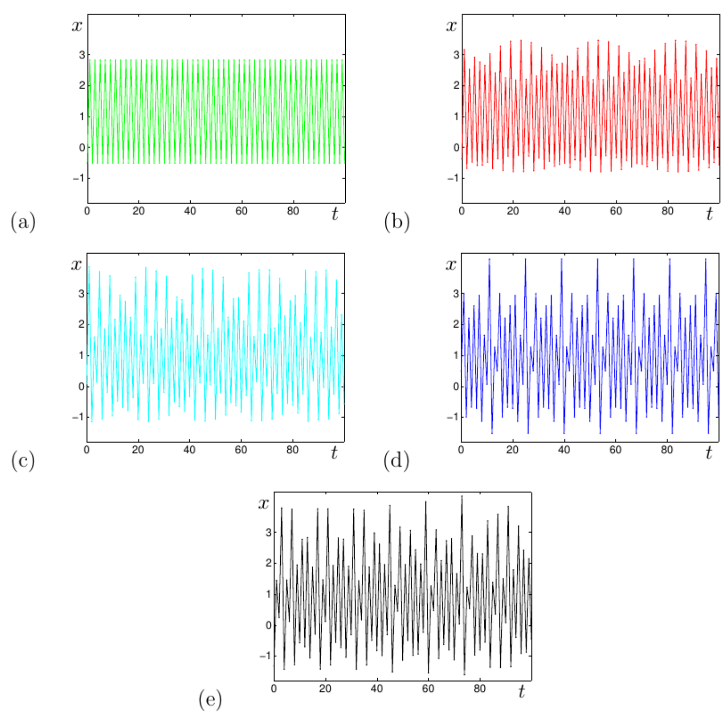

FIGURE 5. Time series of system (2) with $\gamma = -0.7$ for (a) $\sigma = 0.08$, (b) $\sigma = 0.1$, (c) $\sigma = 0.15$, (d) $\sigma = 0.2$, (e) $\sigma = 0.21$.

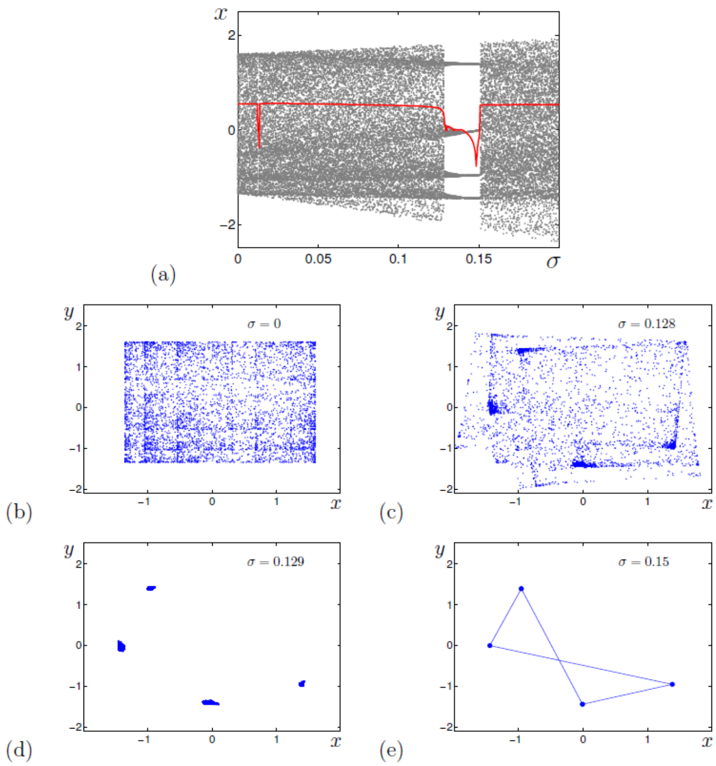

FIGURE 6. System (2) with $\gamma = -2.5$: (a) bifurcation diagram (grey) and Lyapunov exponents (red); attractors for (b) $\sigma = 0$, (c) $\sigma = 0.128$, (d) $\sigma = 0.129$, (e) $\sigma = 0.15$.

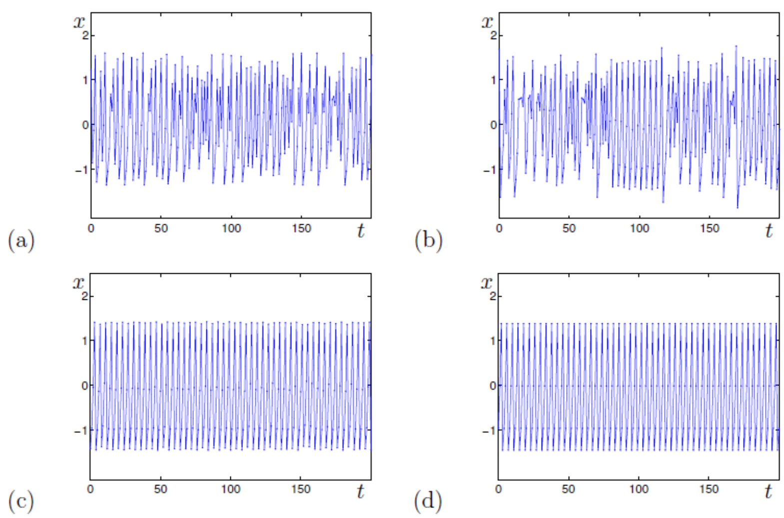

FIGURE 7. Time series of system (2) with $\gamma = -2.5$ for (a) $\sigma = 0$, (b) $\sigma = 0.128$, (c) $\sigma = 0.129$, (d) $\sigma = 0.15$.

with the quasiperiodic attractors. With further increase of the coupling parameter σ, after a series of bifurcations, the quasiperiodic regime transforms back to 2-cycle. Between zones of the quasiperiodicity, "windows" of periodic cycles are observed. These changes of dynamics are also illustrated by the corresponding Lyapunov exponents $\Lambda_1(\sigma)$ (red dashed) and $\Lambda_2(\sigma)$ (blue dashed). As one can see, in this σ-zone, dynamics of coupled neurons essentially depends

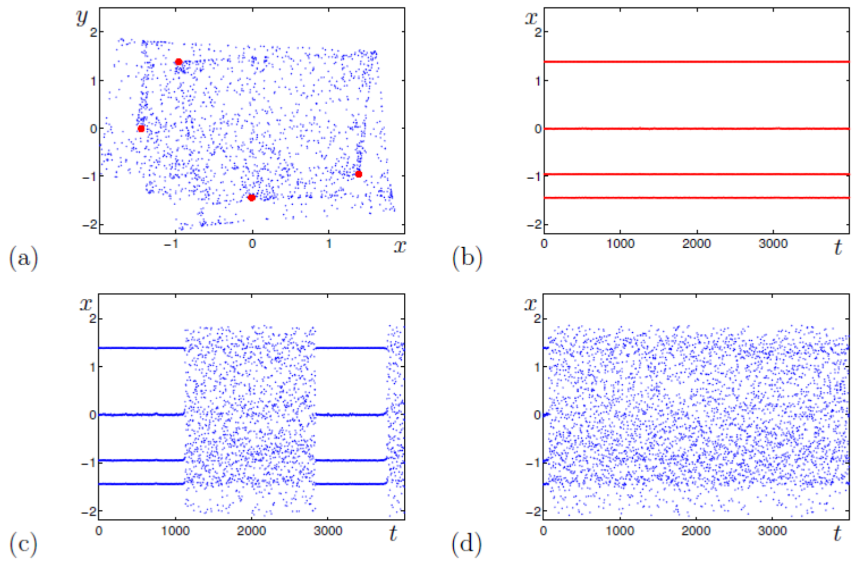

FIGURE 8. Dynamics of stochastic system (3) with $\gamma = -2.5$, $\sigma = 0.15$: (a) random states for $\varepsilon = 0.0005$ (red), $\varepsilon = 0.001$ (blue); time series for (b) $\varepsilon = 0.0005$, (c) $\varepsilon = 0.001$, and (d) $\varepsilon = 0.003$.

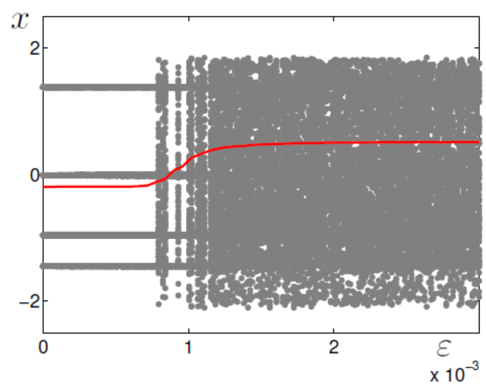

FIGURE 9. Random states (grey) and Lyapunov exponents (red) of stochastic system (3) with $\gamma = -2.5$.

on the initial state.

A further decrease of the parameter γ complicates the structure of coexisting attractors (see Figure 3). Here, we can see how the 2-cycle C_1 (red) splits into 4-cycle whereas the coexisting attractor (blue) exhibits chaotic fragments. This fact is justified by the positiveness of the Lyapunov exponent (blue) in the right panel of Figure 3.

So, due to coupling, two regular oscillating neurons can demonstrate bistable dynamics where regular oscillations coexist with chaotic ones. In Figure 4 (left panel), it is shown how attractors plotted by blue in Figure 4 change as σ increases. For $\sigma = 0$, states of 2-cycle are shown by two black circles. For $\sigma = 0.08$, states of 2-cycle (green) slightly shift. After Neimark-Sacker bifurcation, this attractor transforms into closed curves (for $\sigma = 0.1$, red). For $\sigma = 0.15$ (light blue), a form of these curves becomes more complicated. With further increase of σ, these curves a destroyed. For $\sigma = 0.2$ (blue), system (2) exhibits the discrete 14-cycle. A coexistence of the chaotic attractor (black) with the regular 4-cycle (red) is shown in Figure 4 (right panel) for $\sigma = 0.21$. Time series of attractors considered in Figure 4 are shown in Figure 5 by the same color.

Remember, that isolated neurons exhibited a simple regular dynamics with 2-cycles as attractors. Due to coupling, the system of connected neurons demonstrates a non-trivial dynamics with the rich variety of coexisting attractors, both regular and chaotic.

Consider now a case when isolated neurons are chaotic (see Figure 6 for $\gamma = -2.5$). Attractors (grey) and Lyapunov exponents (red) are shown in Figure 6a versus parameter σ. As one can see, in this σ-region, there is a rather wide window where the coupled neuron system exhibits a simple regular dynamics with 4-cycles as attractors. In Figs. 6b-e, some snapshots of the transformation of the initial chaotic attractor ($\sigma = 0$) into regular one ($\sigma = 0.15$) are shown.

For $\sigma = 0$, the Lyapunov exponent is $\Lambda(0) = 0.53$. For $\sigma = 0.128$, one can see how states of the chaotic attractor begin to concentrate near some four points ($\Lambda(0.128) = 0.35$). For $\sigma = 0.129$, all states of the attractor are localized near these four points, but the attractor is still chaotic with $\Lambda(0.129) = 0.097$. For $\sigma = 0.15$, the system possesses the stable 4-cycle with the negative $\Lambda(0.15) = -0.19$. Corresponding time series are shown in Figure 7 for the same values of the parameter σ.

NOISE-INDUCED EFFECTS IN SYSTEM OF COUPLED NEURONS

Consider how a random variation of the coupling parameter changes the system (2) dynamics. For this study, we will use the following mathematical model

$$\begin{aligned} x_{t+1} &= f(\gamma, x_t) + (\sigma + \varepsilon\xi_t)(y_t - x_t), \\ y_{t+1} &= f(\gamma, y_t) + (\sigma + \varepsilon\xi_t)(x_t - y_t), \end{aligned} \tag{3}$$

where ξ_t is uncorrelated Gaussian noise with parameters $\langle \xi_t \rangle = 0$, $\langle \xi_t^2 \rangle = 1$, and ε is the noise intensity.

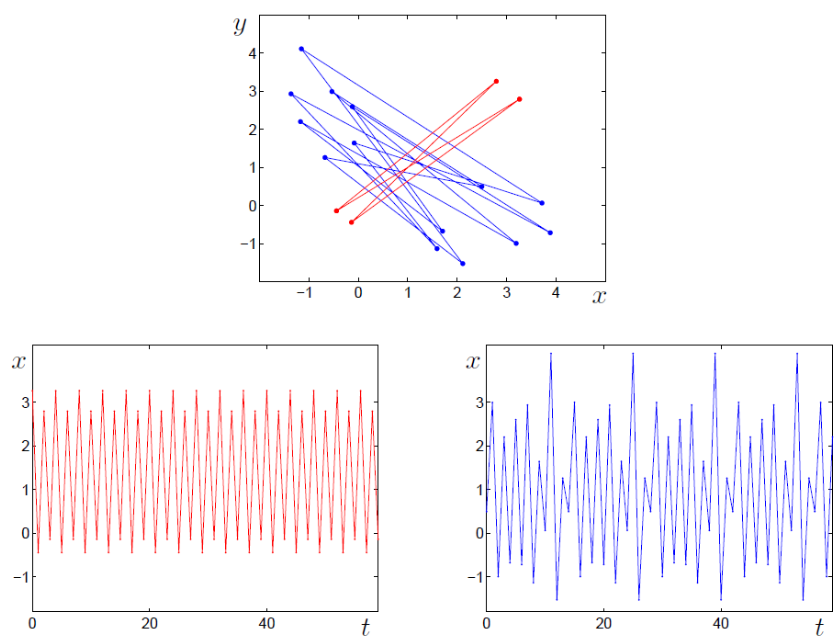

FIGURE 10. Attractors and corresponding time series of system (2) with $\gamma = -0.7$, $\sigma = 0.2$.

Let us fix $\gamma = -2.5$, $\sigma = 0.15$. For this set of parameters, deterministic system (2) exhibits the ordering of dynamics of the coupled neurons which are chaotic in the isolated case (see Fig. 7).

In Figure 8a, random states of system (3) are plotted for two values of the noise intensity. For $\varepsilon = 0.0005$ (red), random states are slightly dispersed near four points of the deterministic 4-cycle. For $\varepsilon = 0.001$ (blue), random states are widely scattered in (x, y)-plane. Such a noise-induced transformation is illustrated by time series in Figs. 8b,c,d. Here, the transient regime with the intermittency of regular and chaotic oscillations can be observed for $\varepsilon = 0.001$.

Details of this noise-induced transformation are shown in Figure 9 where random states (grey) and Lyapunov exponents (red) are plotted versus noise intensity ε. Here, till $\varepsilon \approx 0.0007$, regular noisy oscillations near 4-cycle are

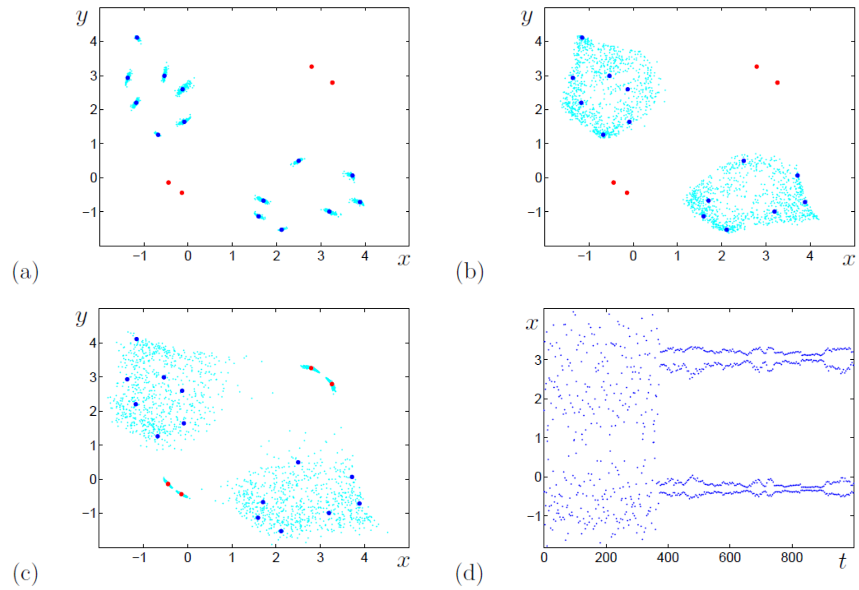

FIGURE 11. Random states of system (3) with $\gamma = -0.7$, $\sigma = 0.2$ and (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.01$, (c) $\varepsilon = 0.05$, (d) time series for $\varepsilon = 0.05$.

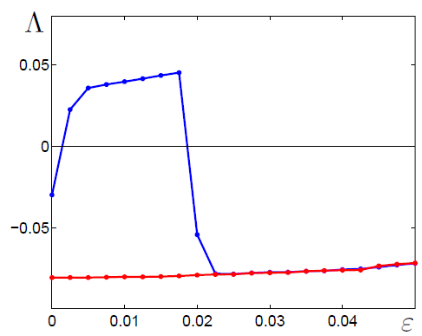

FIGURE 12. Lyapunov exponents with $\gamma = -0.7$, $\sigma = 0.2$ for solutions of system (3) starting from 14-cycle (blue) and 4-cycle (red).

observed. For $\varepsilon > 0.0012$, the system exhibits chaotic oscillations. Between these intervals, the transient dynamics with the intermittency of regular and chaotic phases can be seen. This transformation is accompanied by the change of the sign of the Lyapunov exponents from minus to plus.

Consider now $\gamma = -0.7$, $\sigma = 0.2$. For these parameters (see Fig. 10), deterministic system (2) is bistable with the coexisting 4-cycle (red color) and 14-cycle (blue color). Consider how noise changes dynamics of this system. In Figure 11, solutions starting from the deterministic 14-cycle are shown. For weak noise, random states are localized near points of 14-cycle (see Fig. 11a). Under increasing noise, an essential blurring of the random distribution can be seen (Fig. 11b). For stronger noise with intensity $\varepsilon = 0.05$, random solutions fall into the basin of attraction of 4-cycle and oscillate near its states (see Fig. 11c,d). Such transformations of the spatial distributions are accompanied by the change of the intrinsic dynamic properties. Such changes are quantitatively described by largest Lyapunov exponents Λ. In Figure 12, plots of $\Lambda(\varepsilon)$ are shown for solution starting from 14-cycle (blue) and 4-cycle (red). As can be seen, under increasing noise, stochastic system (3) undergoes transitions ”order-chaos-order”. Moreover, noise transforms the system (3) from bistable regime to monostable one.

Conclusion

In the present paper, a corporate dynamics of two coupled neurons was studied. Two-dimensional dynamic model based on the Rulkov map is used. For isolated neurons, this model demonstrates a diversity of dynamical regimes: equilibrium, periodic and chaotic oscillations. It was shown how these regimes can be transformed in the system with coupling. Due to coupling, the equilibrium regime can transform into oscillatory one, the periodic can transform into quasiperiodic and chaotic, and chaotic regime can become regular with periodic oscillations. Moreover, due to coupling, the system becomes bistable. Noise-induced transformations of these regimes are discussed. Noise-induced transitions "order-chaos" and "order-chaos-order" are demonstrated.

Acknowledgments

The work was supported by Russian Science Foundation (N 16-11-10098).

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